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Tensor product of the octonionic Hilbert spaces and colour confinement

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Abstract. The definition of the tensor product of the octonionic Hilbert spaces with complex geometry is proposed. This definition is based on the isomorphism (geometric and algebraic) of the octonionic Hilbert space with appropriate structure. It is found that the algebraic colour confinement holds only partially. In so called essentially octonionic theories the algebraic confinement of colour holds for all boson states.

1. Introduction

A theoretical description of the leptons and the quark structure of hadrons has been proposed by Günaydin and Gürsey (1973, 1974), Gürsey (1976), Gürsey *et al* (1976), Gürsey and Sikivie (1976), (see also Bucella *et al* 1977) in the context of octonionic quantum mechanics. Earlier Pais (1961) (see also Tiomno 1963) had emphasised the applicability of the octonionic algebra to the classification of the elementary particles. The structure and properties of the Hilbert spaces defined over a non-associative Cayley algebra has been investigated by Goldstine and Horwitz (1964, 1966) and Horwitz and Biedenharn (1965). In the works of Günaydin and Gürsey the spaces of one, two and three particle states were considered using the octonionic Hilbert space (OHS) with complex scalar product. However a systematic description of the multi-particle states is still lacking. In this work we propose the definition of the tensor product of OHS which preserves the octonionic structure. In order to do this we first show that the OHS with complex geometry is isomorphic to the complex Hilbert space (CHS) with appropriate structure. Using this isomorphism we find that the structure of the OHS is essentially determined by the representations $\underset{\sim}{1}$, $\underset{\sim}{4}$ and $\bar{\underset{\sim}{4}}$ of the group $U(4)_c \supset SU(3)_c$. The construction of the multilinear tensor product is based on this fact. The results obtained suggest that in general colour confinement cannot have an algebraic origin. However in the cases when the quark representation of the symmetry group G subduced to $U(4)_c$ contains only the representations $\underset{\sim}{4}$ and $\bar{\underset{\sim}{4}}$ of $U(4)_c$ the complete algebraic confinement of the colour for the boson states holds.

2. Octonionic Hilbert space with complex geometry

In this section we give a short review of the OHS formalism and show the isomorphism between OHS with complex geometry and CHS. In the following we use a real octonion basis $\{e_A\}$ with the multiplication rules $e_0e_A = e_Ae_0 = e_A$ for $A = 0, 1, 2, \ldots, 7$ and

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$e_A e_B = -e_0\delta_{AB} + \Sigma_{ABC} e_C$ for $A, B, C = 1, 2, \ldots, 7$ where $\Sigma_{ijk} = \epsilon_{ijk}$, $\Sigma_{7,k,k+3} = 1$ for $i, k, j = 1, 2, 3$ and $\Sigma_{ABC} = -\Sigma_{BCA} = -\Sigma_{CBA}$. The elements

$$A = e_A a_A = e_0 a_0 + e_1 a_1 + \ldots + e_7 a_7 = e_0 a_0 + e_\alpha a_\alpha,$$

$$A = 0, 1, \ldots, 7, \qquad \alpha = 1, \ldots, 7$$

of the real octonion algebra W can be represented in complex form

$$A = e_\mu A_\mu = e_0 A_0 + \vec{e}\vec{A}, \qquad \mu = 0, 1, 2, 3$$

where the coefficients $A_0 = e_0 a_0 + e_7 a_7$, $A_k = e_0 a_k - e_7 a_{k+3}$ belong to the subset $C(e_0, e_7) \subset W$ isomorphic to the field of the complex numbers. The octonionic algebra is non-associative but the multiplication of the octonions by complex numbers is associative i.e. if $\alpha, \beta \in C$ then $\alpha(\beta A) = (\alpha\beta)A$, $(\alpha A)\beta = \alpha(A\beta)$ for all elements A of W. This fact is critical for the geometry of the OHS. In W we can define octonionic, quaternionic and complex conjugation by

$$\bar{A} \equiv e_0 a_0 - e_\alpha a_\alpha = e_0 A_0^* - \vec{e}\vec{A},$$

$$\tilde{A} \equiv e_0 a_0 - e_k a_k - e_{k+3} a_{k+3} + e_7 a_7 = e_0 A_0 - \vec{e}\vec{A}$$

and

$$A^* \equiv e_0 a_0 + e_k a_k - e_{k+3} a_{k+3} - e_7 a_7 = e_0 A_0^* + \vec{e}\vec{A}^*$$

respectively. Note that only octonionic and complex conjugations are the automorphisms of W. As is well known there are four bilinear forms over W which define the norm $|A| = (a_A a_A)^{1/2} = (A_\mu A_\mu^*)^{1/2}$ with property $|AB| = |A| \cdot |B|$ (see for example Günaydin 1976). In the following we use the complex scalar product defined by

$$\langle A, B\rangle \equiv \tfrac{1}{2}[\bar{A}B + (\widetilde{\bar{A}B})] = A_\mu^* B_\mu.$$

Note that this scalar product is invariant under transformations of the $U(4)$ group. The properties of the complex scalar product are listed in the appendix.

The notion of the OHS with complex geometry can be introduced by little modification of the postulates given by Goldstine and Horwitz (1964).

Postulate 1 (*algebraic*)

$\mathscr{H}$ is a linear vector space over octonions (we adopt a right-handed multiplication convention—for details see appendix). Note that we demand associativity for complex numbers and only power associativity for other scalars.

Postulate 2 (*geometric*)

There exists an inner product (f, g) defined for all f, g in $\mathscr{H}$ with values in C such that

(*a*) $(f, g+h) = (f, g) + (f, h)$

(*b*) $(f, g)^* = (g, f)$

(*c*) $(f, fA) = \frac{1}{2}(A + \tilde{A})|f|^2$

where $|f|^2 = (f, f) \geq 0$ and $|f| = 0$ is equivalent to $f = 0$

(*d*) $(f, \alpha g) = \alpha^*(f, g) \qquad$ for $\alpha \in C$.

These relations imply other useful rules listed in the appendix, in particular the Schwartz inequality holds,

$$|(f, g)| \leq |f| \cdot |g| .$$

Postulate 3 (*topological*)

$\mathscr{H}$ is complete and separable.

From the above postulates it follows that the vector $f \in \mathscr{H}$ can be represented in the form (in complex notation)

$$f = e_\mu f_\mu \tag{1}$$

or in Dirac formalism by $|f\rangle = e_\mu |f_\mu\rangle$, where $f_\mu = \langle e_\mu, f\rangle \in C$. The scalar product has the form

$$(f, g) = \sum_{\mu\nu} (e_\mu f_\mu, e_\nu g_\nu) = \sum_{\mu\nu} \langle e_\mu, e_\nu\rangle (f_\mu, g_\nu) = \sum_\mu (f_\mu, g_\mu) \tag{2}$$

where (f_μ, g_μ) is the standard complex Hilbert space scalar product. Defining the bra vectors by

$$\langle f| \equiv |\bar{f}\rangle^{\mathrm{T}} = \langle f_\mu | \bar{e}_\mu, \qquad \langle f_\mu| = |f_\mu\rangle^{\dagger} = |f_\mu\rangle^{*\mathrm{T}} = |f_\mu^*\rangle^{\mathrm{T}},$$

we can write the scalar product in the form

$$(f, g) \equiv \langle f|g\rangle = \sum_\mu \langle f_\mu | g_\mu\rangle. \tag{3}$$

The closed subset of $\mathscr{H}$, containing together with the vectors f, g all their linear combinations with complex coefficients, will be called the linear manifold. In particular OHS is the direct sum of four orthogonal linear manifolds generated by e_μ. The projectors on the manifolds have standard properties but in general do not commute with multiplication by octonions. This follows from the fact that linear manifolds are in general not closed under multiplication by octonions.

As in the standard (CHS) case $\Pi_f = |f\rangle\langle f|$ is the projection operator on the manifold generated by $|f\rangle$ ($|f| = 1$). The associativity for complex numbers implies that

$$\langle f|g\rangle = \sum_\alpha (\langle f|\alpha\rangle\langle\alpha|)|g\rangle = \sum_\alpha \langle f|(|\alpha\rangle\langle\alpha|g\rangle) = \sum_\alpha (\langle f|\alpha\rangle)(\langle\alpha|g\rangle). \tag{4}$$

The linear operator L is C-linear mapping of the manifold $M \subset \mathscr{H}$ into $\mathscr{H}$. It follows from the associativity of the complex numbers that the operator L can be represented in the orthonormal basis by

$$L|f\rangle = L \sum_\beta |\beta\rangle\langle\beta|f\rangle = \sum_\beta (L|\beta\rangle)\langle\beta|f\rangle = \sum_{\alpha\beta} |\alpha\rangle\langle\alpha|L|\beta\rangle\langle\beta|f\rangle$$

i.e.

$$L = \sum_{\alpha\beta} |\alpha\rangle L_{\alpha\beta}\langle\beta| = \sum_{\alpha\beta} (|\alpha\rangle L_{\alpha\beta})\langle\beta| = \sum_{\alpha\beta} |\alpha\rangle(L_{\alpha\beta}\langle\beta|) \tag{5}$$

where $L_{\alpha\beta} = \langle\alpha|L|\beta\rangle \in C$ (note that in general $|\alpha\rangle L_{\alpha\beta}\langle\beta| \neq L_{\alpha\beta}|\alpha\rangle\langle\beta|$ or $|\alpha\rangle\langle\beta|L_{\alpha\beta}$). Moreover the composition law for linear operators L and N has the standard form

$$NL|f\rangle = \sum_{\alpha\beta\gamma} |\beta\rangle N_{\beta\gamma} L_{\gamma\alpha}\langle\alpha|f\rangle$$

i.e.

$$(NL)_{\beta\alpha} = \sum_\gamma N_{\beta\gamma} L_{\gamma\alpha}. \tag{6}$$

The hermitian and unitary operators can be defined as usual and are represented by hermitian and unitary complex matrices respectively. The projectors on manifolds are hermitian. Note that the hermitian conjugation is not an automorphism of W (but is the automorphism of C) contrary to the standard (complex) case. It is easy to see that if we restrict ourselves to the multiplication of vectors by the elements of C then the geometrical structure of OHS introduced above is isomorphic to the structure of appropriate CHS: with every vector $|f\rangle = e_\mu |f_\mu\rangle \in$ OHS we associate the complex vector

$$f \equiv \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} \in \text{CHS}$$

with scalar product

$$\langle f|g\rangle \equiv (f, g) = \sum_\mu (f_\mu, g_\mu).$$

Similarly, to every linear operator L in OHS there corresponds a linear operator in CHS via relations (5) and (6). The e_0 and e_7 are implemented by 1 and $\mathrm{i} = \sqrt{-1}$ respectively.

Now we show that the CHS mentioned above can be equipped with the algebraic structure of OHS. Let us consider the Cayley group with elements defined by relations $E_{\alpha_1\alpha_2\ldots\alpha_n}|f\rangle \equiv (|f\rangle e_{\alpha_n})\ldots)e_{\alpha_2})e_{\alpha_1}$. If Q is the operator of quaternionic conjugation, i.e. $Q|f\rangle \equiv \widetilde{|f\rangle} = e_0|f_0\rangle - \vec{e}|\vec{f}\rangle$ then the following relations hold:

$$\begin{aligned} &E_k = -\tfrac{1}{2}\epsilon_{kij}E_{123}E_{ij}, \qquad k, i, j = 1, 2, 3 \\ &E_{k+3}|f\rangle = |f\rangle(e_7e_k) = -e_7(|f\rangle e_k) + \{e_7, |f\rangle\}e_k = -(\widetilde{|f\rangle e_k})e_7 + (\widetilde{|f\rangle}e_7)e_k + (|f\rangle e_7)e_k \end{aligned} \tag{7a}$$

i.e.

$$E_{k+3} = (E_k + \{Q, E_k\})E_7 = -QE_kQE_7 \tag{7b}$$

$$E_{\alpha_1\alpha_2\ldots\alpha_n} = E_{\alpha_1}E_{\alpha_2}\ldots E_{\alpha_n}. \tag{7c}$$

From equations ($7a, b, c$) it is evident that the whole Cayley group is generated by $E_0 = I$, E_7, E_{123}, E_{ik} ($i, k = 1, 2, 3$) and Q only.

The multiplication by (e_Ae_B) is represented by

$$(E_i \times E_k)|f\rangle \equiv |f\rangle(e_ie_k) = \epsilon_{ikj}E_j|f\rangle \tag{8a}$$

$$(E_k \times E_k)|f\rangle \equiv |f\rangle(e_ke_k) = -E_0|f\rangle \tag{8b}$$

$$(E_7 \times E_k)|f\rangle \equiv |f\rangle(e_7e_k) = E_{k+3}|f\rangle \qquad i, k = 1, 2, 3 \tag{8c}$$

$$(E_0 \times E_A)|f\rangle \equiv E_A|f\rangle \tag{8d}$$

where $A = 0, 1, 2, \ldots, 7$. The formulae for other products can be obtained from equations ($8a, b, c, d$) and the octonionic multiplication table. From the properties of the octonion algebra and the complex scalar product we obtain useful relations for E_7, E_{123}, E_{ik} and Q:

$$\begin{aligned} &E_{ik} = -E_{ki} = -E_{ik}^{-1} = -E_{ik}^{\dagger}, \quad i \neq k, && E_{123} = E_{231} = -E_{213}, && E_{123}^2 = I, \\ &(E_{123}f, g) = (f, E_{123}g)^{*\dagger}, && E_7 = -E_7^{-1} = -E_7^{\dagger}, && Q = Q^{+} = Q^{-1}, \\ &[E_7, E_{ik}] = 0, && \{E_7, E_{123}\} = 0, && [E_7, Q] = 0. \end{aligned}$$

The correspondence between OHS and CHS implies that the E_0, E_7, E_{ik} and Q are represented as follows

$$E_0 = I, \qquad E_7 = \mathrm{i}I, \qquad Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$E_{12} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \qquad E_{23} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$E_{31} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \tag{9}$$

On the other hand the E_{123} acts as the complex conjugation of the vector

$$f = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}.$$

Thus the OHS is isomorphic (algebraically and geometrically) to the CHS defined above. Now let us explain the role of the $U(4)$ group in the structure of the OHS. As is well known the octonion algebra admits G_2 as the group of automorphisms. On the other hand the scalar product in OHS is invariant under $U(4)$ 'gauge' group. The intersection of these groups is $G_2 \cap U(4) \sim SU(3)$. The action of this common subgroup leaves the algebraic and geometric structure of OHS unaffected; in papers by Günaydin and Gürsey (1973, 1974, 1976) $G_2 \cap U(4)$ is identified with the $SU(3)_c$ colour group. The f_0 and f_k form the colour singlet and triplet respectively.

Now we make an important remark. The algebraic closure (under matrix multiplication) of the unitary, unimodular 4×4 matrices E_{ik}, E_7 and the matrices of the representation $\underline{1} \oplus \underline{3}$ of the colour $SU(3)_c$ group forms the self-representation $\underline{4}$ of the $SU(4)$ group. Furthermore the unitary matrix Q ($\det Q = -1$) extends this group to the subgroup of the $U(4)$ 'gauge' group with $(\det U)^2 = 1$. This group will be denoted by $U(4)_c$. From the above it follows that the algebraic structure of the OHS is essentially determined by four-dimensional self-representation $\underline{4}$ of the $U(4)_c$ group (and in fact its adjoint $\bar{\underline{4}}$ since E_{123}: $\underline{4} \to \bar{\underline{4}}$) because: ($a$) it contains the intrinsic symmetry group $SU(3)_c$; (b) the E_{ik}, E_7 and Q, which belong to the $U(4)_c$, together with the operator of complex conjugation generate the Cayley group via equations (7), (8) and (9).

3. Tensor product of the octonionic Hilbert spaces

In the preceding section it was shown that the OHS is isomorphic to the CHS carrying the self-representation of the group $U(4)_c$. In this CHS the octonionic structure is represented by appropriately defined operations. This result allows us to define in a consistent manner the tensor product of the OHS. It is natural to demand that the resulting Hilbert space has a similar structure, i.e. is the carrier space for simultaneously irreducible representations of the $U(4)_c$ and Cayley groups with a common subgroup containing the elements E_{ik}, E_7 and Q. It is not difficult to show that this condition is fulfilled for scalar ($\underline{1}$) and self-representations ($\underline{4}, \bar{\underline{4}}$) of the group $U(4)_c$ only. For example in the product of even number of $\underline{4}$, $\bar{\underline{4}}$ the E_0 and $-E_0$ are represented by $+I$ and consequently the Cayley representation group is one-dimensional (Goldstine *et al* 1964). Thus $U(4)_c$ acts in this case trivially; this holds only for some even products like $\underline{4} \otimes \bar{\underline{4}} = \underline{15} \oplus \underline{1}$, $\underline{4} \otimes \underline{4} \otimes \underline{4} \otimes \underline{4} = 3 \cdot \underline{45} \oplus \underline{35} \oplus 2 \cdot \underline{20} \oplus 3 \cdot \underline{15} \oplus \underline{1}$ etc. The above considerations suggest the following form for the definition of the tensor product:

$$\mathscr{H} \times \mathscr{H} \times \ldots \times \mathscr{H} \equiv \Pi\,(\mathscr{H} \otimes \mathscr{H} \otimes \ldots \otimes \mathscr{H}) \tag{10}$$

where Π projects on the whole subspace of the representation $(\oplus \underline{1}) \oplus (\oplus \underline{4}) \oplus (\oplus \bar{\underline{4}})$ and $\otimes$ denotes the standard tensor product.

Thus $\mathscr{H} \times \ldots \times \mathscr{H} = (\oplus \mathscr{H}^1) \oplus (\oplus \mathscr{H}^4) \oplus (\oplus \mathscr{H}^{\bar{4}})$ where $\mathscr{H}^4$ and $\mathscr{H}^{\bar{4}}$ are the CHS defined above while the $\mathscr{H}^1$ is $U(4)_c$ scalar. The action of the Cayley group in $\mathscr{H}^1$ is defined by homomorphism $\pm E_0$, $\pm E_7 \rightarrow I$, $\pm E_k$, $\pm E_{k+3} \rightarrow$ operator of the complex conjugation. Note that our definition of the tensor product is almost analogous to the symmetrisation or antisymmetrisation of the multiparticle boson or fermion states respectively. However there is a very important difference because the octonionic tensor product of some number of one-particle OHS cannot be obtained starting from one copy and multiplying successively by others. For example $\mathscr{H}^4 \times \mathscr{H}^4 \times \mathscr{H}^4 = \mathscr{H}^4 \neq \mathscr{H}^4 \times (\mathscr{H}^4 \times \mathscr{H}^4) = 0$ because $\Pi(\underline{4} \otimes \underline{4} \otimes \underline{4}) = \Pi(\underline{3 \cdot 20} \otimes \bar{\underline{4}}) = \bar{\underline{4}}$ whereas $\Pi(\underline{4} \otimes \Pi(\underline{4} \otimes \underline{4})) = 0$.

Let us consider the case when the theory based on OHS formalism possess the symmetry group G. Then G must necessarily contain $U(4)_c$ as the subgroup. The foregoing discussion implies that the only admissible representations D of G fulfil the condition

$$D(G) \downarrow U(4)_c = (\oplus \underline{1}) \oplus (\oplus \underline{4}) \oplus (\oplus \bar{\underline{4}}). \tag{11}$$

Consequently the definition of tensor product should be generalised as follows

$$\mathscr{H}_1 \times \mathscr{H}_2 \times \ldots \times \mathscr{H}_n \equiv \Pi(\mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \ldots \otimes \mathscr{H}_n). \tag{12}$$

Here the $\mathscr{H}_k$'s are the carrier spaces of the admissible (in the sense of equation (11)) representations of G. The operator Π projects the standard tensor product of $\mathscr{H}_k$'s on the whole subspace of the admissible representations of G. The definition (12) implies strong selection rules on the acceptable multiplets of G.

If the (faithful) admissible representations of G are such that subduced to the $U(4)_c$ contain the $\underline{4}$ and $\bar{\underline{4}}$ only, the resulting theory will be called essentially octonionic. Note that in the essentially octonionic theories the algebraic colour confinement for the boson states holds. In fact the octonionic tensor product of an even number of the admissible representations (associated with quarks) is the $U(4)_c$ singlet (or equals zero).

We wish now to discuss the field concept in this framework. As usual the field Φ is an operator valued tempered distribution acting in OHS and its domain is a linear manifold dense in OHS. It can be represented by $\Phi(x) \equiv e_\mu \Phi_\mu(x)$ or equivalently by

$$\Phi(x) \equiv \begin{pmatrix} \Phi_0(x) \\ \Phi_1(x) \\ \Phi_2(x) \\ \Phi_3(x) \end{pmatrix}.$$

The vacuum $|0\rangle$ belongs to the $\mathcal{H}^1$. Note that in general we can generate from the vacuum the vectors belonging rather to the standard tensor product of one-particle states. For this reason it is desirable to define the product of the field operators by

$$\Phi \times \Phi \times \ldots \times \Phi \equiv \Pi(\Phi \otimes \Phi \otimes \ldots \otimes \Phi). \tag{13}$$

Then the vectors $(\Phi \times \Phi \times \ldots \times \Phi)|0\rangle$ belong to the octonionic tensor product of one-particle states. However no set of commutation rules for the field operators can be found which allows us to obtain the above vectors and only those, by successive action of the field operators on vacuum.

Now we give some examples illustrating the above formalism.

Example 1. Let us consider the fictitious theory based on the group $O(7) \supset U(4)_c$. From equation (11) it follows that the only admissible representations of $O(7)$ are scalar $\underset{\sim}{1}$ and spinor $\underset{\sim}{8}$. The $U(4)_c$ content of the eight-dimensional spinor representation of $O(7)$ is given by

$$\underset{\sim}{8} = \underset{\sim}{4} \oplus \underset{\sim}{4}.$$

Therefore the theory based on $O(7)$ is essentially octonionic. If we associate the quarks with representation $\underset{\sim}{8}$ then the boson states are colour and $O(7)$ singlets. On the other hand the fermions can occur in octets only.

Example 2. Let $G = U(4)_c \times G_F$ be the symmetry group of the theory. The admissible representations of G have the form $(\underset{\sim}{4}, D_F)$, $(\underset{\sim}{4}, D_F)$, $(\underset{\sim}{1}, D_F)$ where D_F denotes an arbitrary representation of G_F. If the quarks are identified with the representation $(\underset{\sim}{4}, \mathcal{D})$ where $\mathcal{D}$ is the fundamental representation of G_F then the resulting theory will be essentially octonionic. Consequently the boson states are colour singlets (but in general not the singlets of G). Note that the diquark states do not exist because $\underset{\sim}{4} \times \underset{\sim}{4} = \Pi(\underset{\sim}{4} \otimes \underset{\sim}{4}) = 0$.

A natural question arises as to how our formalism is connected with the coloured quarks scheme of Günaydin and Gürsey (1973). It is easy to see that their model can be obtained (in the framework of example 2) by breaking the $U(4)_c$ group down to $SU(3)_c$ by the condition $q_0 = 0$. Here $q_\mu(x)$, $\mu = 0, 1, 2, 3$, denotes the quark field (the flavour indices are omitted). However, as it follows from our considerations, the conclusion of Günaydin and Gürsey that their scheme is a realisation of the proposal of Gell-Mann (1972) with natural algebraic confinement of quarks, is false (see also below).

4. Conclusions

We start from a review of the arguments given by Günaydin and Gürsey (1974, 1976).

First we note that there is no mixing between colour singlets and triplets because the $SU(3)_c$ symmetry is exact. Therefore the physical superposition of the physical states in OHS has the C-number coefficients (the superposition with octonionic coefficients mixes, in general, singlets and triplets). So every subset of physical states generates the linear manifold of physical states in OHS, i.e. the superposition principle does not conflict with the geometry of OHS. Concluding the Birkhoff–von Neumann propositional calculus remains unaffected. Our results contradict the claim by Günaydin and Gürsey that the colour subspace in OHS is confined. Their arguments are based on the statement that the Birkhoff–von Neumann propositional calculus cannot be realised in whole OHS and there is no satisfactory way of defining tensor product states. This is true in cases of the OHS with octonionic or quaternionic geometry. However the above considerations show that it is false in case of the OHS with complex geometry.

Let us summarise the results.

(1) The quantum theory based on the OHS with complex geometry can be treated as the standard quantum theory with appropriate structure and selection rules.

(2) The algebraic colour confinement can hold for the boson states in essentially octonionic theories. For fermions the confinement, if it exists, is of dynamical nature.

(3) The examples given in § 3 indicate that there are very strong restrictions on the admissible multiquark states.

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Appendix 1. The properties of the complex scalar product in the octonion algebra $\boldsymbol{W}(\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{W}, \boldsymbol{\alpha}, \boldsymbol{\beta} \in \boldsymbol{C} \subset \boldsymbol{W})$

$$\langle A, B\rangle^* = \langle B, A\rangle = \langle A^*, B^*\rangle, \qquad \langle \alpha A, B\rangle = \langle A, \alpha^* B\rangle$$

$$\langle A\alpha, B\rangle = \langle A, B\alpha^*\rangle = \alpha^*\langle A, B\rangle$$

$$\langle A, A\rangle = |A|^2 \in \mathbb{R}_+, \qquad |A| = 0 \Leftrightarrow A = 0$$

$$\langle A, B + C\rangle = \langle A, B\rangle + \langle A, C\rangle, \qquad \langle AB, AB\rangle = |A|^2 \,.\, |B|^2$$

$$\langle A, AB\rangle = \tfrac{1}{2}(B + \tilde{B})|A|^2, \qquad \langle AB, A\rangle = \tfrac{1}{2}(B + \tilde{B})^* |A|^2$$

$$\langle e_\mu, e_\nu\rangle = \delta_{\mu\nu}, \qquad \langle e_i, e_{k+3}\rangle = -e_7\delta_{ik}, \qquad \langle e_0, e_7\rangle = e_7$$

$$\langle e_{i+3}, e_{k+3}\rangle = \delta_{ik}, \qquad \langle e_0, e_{k+3}\rangle = \langle e_7, e_{k+3}\rangle = \langle e_7, e_k\rangle = 0$$

Appendix 2. The algebraic postulate of the OHS

(1) $\mathscr{H}$ is an additive abelian group.

(2) The mapping $\mathscr{H} \times W \to \mathscr{H}$ is defined which satisfies ($f, g \in \mathscr{H}$, $A, B \in W$, $\alpha, \beta \in C \subset W$)

(*a*) distributive laws

$$(f + g)A = fA + gA$$

$$f(A + B) = fA + fB$$

(*b*) associativity for complex numbers

$$(f\alpha)\beta = f(\alpha\beta)$$

(*c*) power associativity

$$(fA)A = fA^2$$

$$(fA)A^{-1} = f$$

(*d*) $f \,.\, 1 = f$.

Appendix 3. The properties of the scalar product in the OHS

$$(f, g)^* = (g, f) = (f^*, g^*), \qquad (f, g^*) = (g, f^*)$$

$$(f\alpha, g) = (f, g\alpha^*) = \alpha^*(f, g)$$

$$(\alpha f, g) = (f, \alpha^* g) \neq \alpha(f, g) \text{ (in general)}, \qquad \alpha \in C$$

$$(f, fA) = \tfrac{1}{2}(A + \tilde{A})|f|^2, \qquad (fA, f) = \tfrac{1}{2}(A + \tilde{A})^*|f|^2$$

$$(fA, fA) = |f|^2 \,.\, |A|^2, \qquad (e_\mu f, e_\nu g) = \delta_{\mu\nu}(f, g).$$

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